14.4 LINEAR APPROXIMATION.

1P,

(I) Tangent plane for

$$f(x,y) = e^{-x^2} + e^{-4y^2}$$ MAPLE Worksheet.

at $(-1, 0, e^{-1}+1)$

$f_x = -2x e^{-x^2}$, $f_y = -8y e^{-4y^2}$

$f_x(-1,0) = +2e^{-1} = +\frac{2}{e}$, $f_y(-1,0) = 0$

$\therefore$ $\Pi$: $z = f(x_0,y_0) + f_x(\underset{-1,0}{x_0,y_0})(x-x_0) + f_y(\underset{-1,0}{x_0,y_0})(y-y_0)$

or $z = e^{-1}+1+2e^{-1}(x+1)+0(y-0) = e^{-1} + \frac{2}{e}(x+1)+1$

or $\boxed{z - \frac{2}{e}x = 1+e^{-1}+2e^{-1} = 1+3e^{-1}}$

(Ask for normal:) normal $= \vec{n} = \langle -\frac{2}{e}, 0, 1 \rangle$.

$\boxed{z = \frac{2}{e}x + 1 + \frac{3}{e}}$ Equation of plane.

(II) Linear Approximation of $f(x,y) = e^{-x^2} + e^{-4y^2}$ at $(-1,0)$.

$\boxed{L(x,y) = \frac{2}{e}x + 1 + \frac{3}{e}}$ or tangent plane approximation of $f$ at $(-1,0)$.

How good is this approximation in the vicinity of $(-1,0)$ ?

<u>answ:</u> $L(-1.1,\, 0.1) \simeq 1.29$

$f(-1.1,\, 0.1) \simeq 1.26$

$3_T'$

## 0.7 Definition 6: Tangent plane

- Consider a function $f(x, y)$ defined in $\mathcal{D}$ that represents a surface $S$ in $\mathbb{R}^3$.

- This function has continuous first partial derivatives in $\mathcal{D}$.

- Also, $P_0(x_0, y_0, z_0)$ is the center of an open disc $B_r(P_0) \subseteq \mathcal{D}$.

- Let $C_1$ and $C_2$ be the two curves obtained by intersecting the surface $S$ with the two planes $x = x_0$ and $y = y_0$, respectively. If $T_1$ and $T_2$ are the tangent lines at the point $P_0$ to the curves $C_1$ and $C_2$, respectively,

then the tangent plane to the surface $S$ at $P_0$ is defined to be the plane that contains both tangent lines $T_1$ and $T_2$.

## 0.8 Theorem 2: Equation of the tangent plane

If $f(x, y)$ and the surface $S$ satisfy the conditions of the previous definition then, an equation of the tangent plane to the surface $S$, represented by the equation $z = f(x, y)$, at the point $P_0$ is given by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

## 0.9 Definition 7: Linear approximation of $f$ at $(x_0, y_0)$

For $f$ satisfying the conditions of Definition 1, we define the *linear approximation of $f$ at* $(x_0, y_0)$ as

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

## 0.10 Definition 8: Differentiability

Assume $f$ has partial derivatives at $(x_0, y_0)) \in \mathcal{D}$, we will say that $f$ is differentiable at $(x_0, y_0)$ if

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $\epsilon_1, \ \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$

Briefly
Set this
thm
and immediately

## 0.11 Theorem 3: Sufficient condition for differentiability

If the partial derivatives $f_x$ and $f_y$ exist near $(x_0, y_0)$ and are continuous at $(x_0, y_0)$, then $f$ is differentiable at $(x_0, y_0)$.

#45) If $f(x,y)$ is differentiable at $(a,b)$ then $f$ is Conts in $(a,b)$.

<u>Proof.</u>

$f$ differentiable at $(a,b)$ means (definition).

$$f(a+\Delta x, b+\Delta y) - f(a,b) =$$

$$f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

What happens when $(\Delta x, \Delta y) \to (0,0)$ ?

$$f(a+\Delta x, b+\Delta y) - f(a,b) \longrightarrow 0$$

or $$\lim_{(\Delta x,\Delta y)\to(0,0)} f(a+\Delta x, b+\Delta y) = f(a,b).$$

If $x = a+\Delta x$, $y = b+\Delta y$

then, $$(x,y) = (a+\Delta x, b+\Delta y) \xrightarrow[(\Delta x,\Delta y)\to(0,0)]{} (a,b)$$

and $$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(\Delta x,\Delta y)\to(0,0)} f(a+\Delta x, b+\Delta y) = f(a,b).$$

Therefore, $f$ is continuous at $(x,y)=(a,b)$.

Turn

$3^{2}_{1}$

## 0.12 Definition 8: Total differential

For $f$ differentiable on $\mathcal{D}$, we define the total differential function as

$$dz(dx, dy) = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

In particular, if $dx = \Delta x = x - x_0$ and $dy = \Delta y = y - y_0$ then,

$$dz(\Delta x, \Delta y) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

## 0.13 Definition 9: Directional Derivatives

Consider a function $f(x, y)$ defined in $\mathcal{D}$ that represents a surface $S$ in $\mathbb{R}^3$. Also, $P_0(x_0, y_0, z_0)$ is the center of an open disc $B_r(P_0) \subseteq \mathcal{D}$. Then, we define the directional derivative of $f$ at $(x_0, y_0)$ in the direction of the vector $\mathbf{u} = \langle a, b \rangle$ as

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

## 0.14 Theorem 4

If $f$ is a differentiable function of $x$ and $y$ in a domain $\mathcal{D}$, then $f$ has directional derivatives in the direction of any unit vectors $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$$

## 0.15 Definition 10: Gradient

If $f$ has partial derivatives on $\mathcal{D}$, the gradient of $f$ is the vector function $\nabla f$ defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

## 0.16 Proposition 1

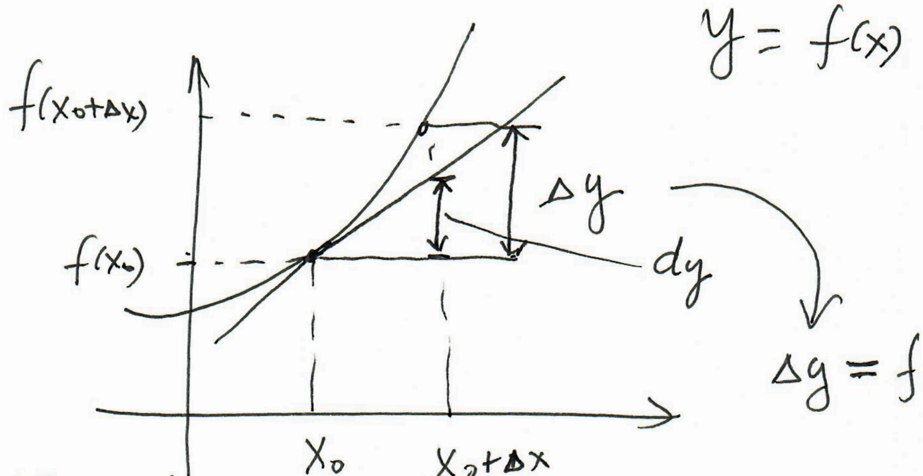
If $f$ is a differentiable function of $x$ and $y$ in a domain $\mathcal{D}$, then the directional derivative of $f$ in the direction of the unit vector $\mathbf{u} = \langle a, b \rangle$ can be written as

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

# Total Differential

Consider $f: D \subset \mathbb{R} \to \mathbb{R}$

$x \to f(x)$




$\Delta y = f(x_0+\Delta x) - f(x_0)$

Tangent line

$$L(x) = f(x_0) + f'(x_0)(x - x_0).$$

then $L(x_0+\Delta x) = f(x_0) + f'(x_0)\Delta x.$

there $$dy = L(x_0+\Delta x) - L(x_0) = f'(x_0)\Delta x.$$

and $$\Delta y = f(x_0+\Delta x) - f(x_0)$$

for

$$f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x, y) \longrightarrow f(x, y)$$

$$z = f(x, y)$$

$$\boxed{\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).}$$

Tangent plane:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

then,

$$L(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

then

$$dz = L(x_0 + \Delta x, y_0 + \Delta y) - L(x_0, y_0)$$

or

$$\boxed{dz = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.}$$

## Concept of Differential. Motivation

14.4 #34)

Estimate amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

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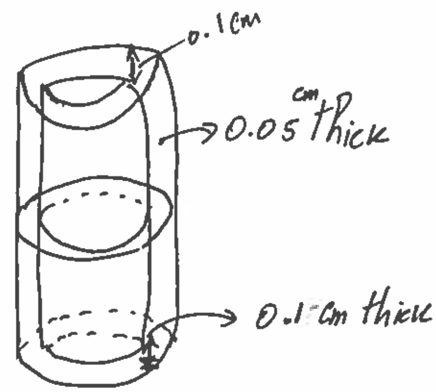
answ:




Amount of metal?

What to do?

Strategy: compute two volumes and subtract.

$$V(r,h) = \pi r^2 h. \quad , \quad r = 2 \text{ cm}. \quad h = 10 \text{ cm}.$$

$$V(2.05, 10.1) - V(2,10) = \text{Exact}.$$

To estimate we use the linear approximation centered at $(2,10)$.

$$L(r,h) = V(2,10) + V_r(2,10)(r-2) + V_h(2,10)(h-10).$$

Notice that

$$L(2,10) = V(2,10)$$

Then

$$\underbrace{L(2.05, 10.2) - L(2,10)}_{\text{This difference is called } dv} = \underbrace{V_r(2,10)(0.05) + V_h(2,10)(0.2)}_{\text{This is call total differential of } V(r,h) \text{ for } dr=0.05 \text{ and } dh=0.1}$$

or $$dv = \frac{\partial v}{\partial r}(2,10)(0.05) + \frac{\partial v}{\partial h}(2,10)(0.2)$$

Now, $$\frac{\partial v}{\partial r} = 2\pi r h, \quad \frac{\partial v}{\partial h}(r,h) = \pi r^2$$

So $$\frac{\partial v}{\partial r}(2,10) = 40\pi, \quad \frac{\partial v}{\partial h}(2,10) = 4\pi$$

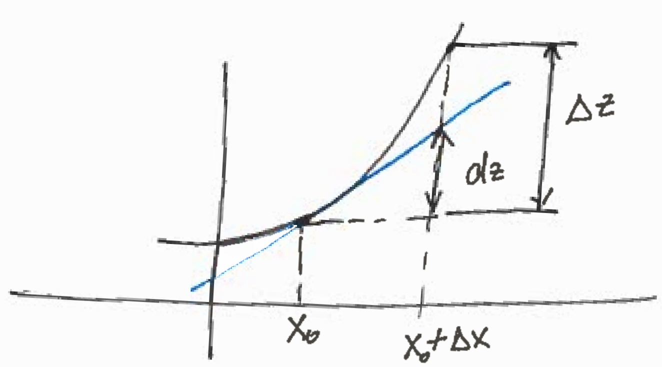
Thus,
$$dv = 40\pi(0.05) + (4\pi)(0.2) \simeq 2.80\pi$$
$$\simeq 8.8\ cm^3.$$

In general,

For a differentiable function $f$ on $D$, the total differential is defined as

$$\boxed{dz(dx, dy) = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy.}$$

Geometrical Interpretation: (1-D) Similar in higher dimensions.




So, $$dz = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$$

and

$$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$$